

NONDENSITY OF ORBITAL SHADOWING PROPERTY IN C^1 -TOPOLOGY.

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The orbital shadowing property (OSP) of discrete dynamical systems on smooth closed manifolds is considered. Nondensity of OSP with respect to the C^1 -topology is proved. The proof uses the method of skew products developed by Yu.S. Ilyashenko and A.S. Gorodetski.

§1. Introduction

The theory of shadowing studies the problem of closeness of approximate and exact trajectories (or orbits) of dynamical systems on unbounded time intervals. This problem is important both for applications (as a rule, approximate trajectories generated by computer simulation of a system are considered) and for the qualitative theory of dynamical systems (shadowing properties can be considered as weak forms of structural stability). Note that we consider only discrete-time dynamical systems (cascades) generated by homeomorphisms of metric spaces and diffeomorphisms of closed smooth manifolds. In this paper, we do not distinguish between a homeomorphism and the dynamical system generated by this homeomorphism. Roughly speaking, a cascade has one of the shadowing properties if any "sufficiently precise" approximate trajectory is "close" to an exact one. Since the statement that approximate trajectories (pseudotrajectories) and exact trajectories are close can be formalized in various ways, there are several shadowing properties. Let us mention the pseudo orbit tracing property POTP, the orbital shadowing property OSP, and the weak shadowing property WSP. The state of the art of the theory of shadowing is described in the monographs [11, 10]. Let us give exact definitions of the shadowing properties that are used in this paper.

Let f be a homeomorphism of a metric space M with metric dist . Let us recall the definitions of the exact trajectory of a point $p \in M$ of the homeomorphism f and its positive and negative semitrajectories:

$$O(p, f) = \left\{ f^k(p) \mid k \in \mathbf{Z} \right\},$$

$$O_+(p, f) = \left\{ f^k(p) \mid k \in \mathbf{Z}, k \geq 0 \right\},$$

$$O_-(p, f) = \left\{ f^k(p) \mid k \in \mathbf{Z}, k \leq 0 \right\}$$

(hereinafter, we denote by \mathbf{Z} the set of integers).

For convenience, without additional remarks, we often use the notation

$$p_k = f^k(p) \quad \text{for } k \in \mathbf{Z}.$$

In addition, we sometimes identify a periodic point with its trajectory, i.e. the set $O(p, f)$.

As usual, we say that a sequence $\xi = \{x_k\} \subset M$ is a d -pseudotrajectory if

$$\text{dist}(x_{k+1}, f(x_k)) < d \quad \text{for } k \in \mathbf{Z}.$$

Thus, a d -pseudotrajectory is one of possible formalizations of the notion of an approximate trajectory.

We say that the homeomorphism f of the space M has POTP (pseudo orbit tracing property) if for any $\epsilon > 0$ there exists a d such that for any d -pseudotrajectory $\xi = \{x_k\}$ one can find a point $q \in M$ such that

$$\text{dist}(x_k, f^k(q)) < \epsilon \quad \text{for } k \in \mathbf{Z}.$$

In other words, POTP means that any "sufficiently precise" approximate trajectory is shadowed by an exact trajectory (i.e. is pointwise close to it).

By $N(\epsilon, A)$ denote the ϵ -neighborhood of a set $A \subset M$. In the paper [14], definitions of the orbital shadowing property (OSP) and the weak shadowing property (WSP) are given. We say that the homeomorphism f of the space M has OSP and write $f \in \text{OSP}$ if for any $\epsilon > 0$ there exists a $d > 0$ such that for any d -pseudotrajectory ξ one can find a point $q \in M$ such that

$$\xi \subset N(\epsilon, O(q, f)) \quad \text{and} \quad O(q, f) \subset N(\epsilon, \xi). \quad (1.1)$$

We say that the homeomorphism f of the space M has WSP if for any $\epsilon > 0$ there exists a $d > 0$ such that for any d -pseudotrajectory ξ one can find a point $q \in M$ such that

$$\xi \subset N(\epsilon, O(q, f)).$$

OSP is a weak analog of POTP: the difference is that we do not require a point x_k of a pseudotrajectory $\xi = \{x_k\}$ and the point $f^k(q)$ of an exact trajectory $O(q, f)$ to be close "at any time moment", instead, the sets of the points of the pseudotrajectory ξ and the trajectory $O(q, f)$ are required to be close. The weak shadowing property WSP is a weak variant of OSP: the difference is that a set of points of a "sufficiently precise" pseudotrajectory ξ is required to be contained in a small neighborhood of some exact trajectory $O(q, f)$.

Let M be a closed smooth manifold. As usual, denote by $\text{Diff}^1(M)$ the set of diffeomorphisms of the manifold M with the C^1 -topology (cf., e.g., [12] for definition).

Generic properties are objects of a special interest in the theory of dynamical systems. We say that a property is generic if it holds for all cascades from a Baire second category set (cf., e.g., [12] for definition) in a space of dynamical systems with a certain topology, and we say that a property is dense if it holds for all cascades from a dense set. In the paper [13], S. Yu. Pilyugin and O. B. Plamenevskaya proved the genericity of POTP with respect to the C^0 -topology if the phase space is a closed smooth manifold (hereinafter, we consider, precisely, this case). The genericity of POTP with respect to the C^0 -topology implies the C^0 -genericity of OSP and WSP. Ch. Bonatti, L. J. Diaz and G. Turcat [1] proved that POTP is nondense with respect to the C^1 -topology, and S. Crovisier [2] proved that WSP is C^1 -dense (also, cf., the paper of S. Yu. Pilyugin, K. Sakai and O. A. Tarakanov [15]).

Our main goal is to prove the C^1 -nondensity (and, therefore, the C^1 -nongenericity) of OSP, which takes the "intermediate" position between WSP and POTP. As usual, denote by S^2 the two-dimensional sphere and by S^1 the circle. Our main result is the following theorem:

Theorem A. *There exists a domain $W \subset \text{Diff}^1(S^2 \times S^1)$ such that any diffeomorphism $f \in W$ does not have OSP.*

In order to prove it, we use an idea originating in works of A. S. Gorodetski and Yu. S. Ilyashenko: to construct the example in a class of partially hyperbolic skew products. To be precise, we consider a certain step skew product G_0 over the Bernoulli shift σ with the fibre homeomorphic to the circle (all necessary definitions are given later). Having realized the Bernoulli shift as a mapping of the Smale horseshoe that is sufficiently fast contracting and expanding compared with the fibre dynamics, we see that a local maximal partially hyperbolic set with center fibres homeomorphic to the circle corresponds to this skew product. Furthermore, the technique of Hirsch-Pugh-Shub-Gorodetski (cf. [6, 7, 3]) implies that the partially hyperbolic set persists under small perturbations of this smooth realization remaining the product of the circle and the Smale horseshoe. And, due to Holder dependence of (individually smooth) center fibres on the point in the base (i.e., in the Smale horseshoe), this skew product is a Holder mild skew product.

We take a sufficiently small C^1 -neighborhood of the step skew product G_0 as the required neighborhood W from Theorem A. In particular, the neighborhood is chosen so small that any diffeomorphism from W is assigned to some mild skew product. Further, we show that Theorem A can be reduced to Theorem A' (which will be exactly formulated in the next section).

Theorem A'. *Any Holder mild skew product "sufficiently close" to the skew product G_0 does not have OSP.*

The proof of Theorem A' is split into two cases. The first case (Case (A1)) corresponds to the situation when there exist two hyperbolic periodic points p (with the one-dimensional unstable manifold) and q (with the one-dimensional stable manifold) such that this manifolds intersect. In this case, using Main Lemma, we construct a pseudotrajectory that can not be orbitally shadowed by any exact trajectory.

The second case (Case (A2)) corresponds to the situation when there are no such intersections. In this case, we construct a pseudotrajectory such that any exact trajectory that orbitally shadows the pseudotrajectory should be the heteroclinic trajectory connecting two hyperbolic periodic points with the one-dimensional unstable and the one-dimensional stable manifolds, respectively. The assumption that the cascade has OSP contradicts to the assumption about the absense of such intersections.

Let us describe the further structure of the paper. In Sec. 2, main definitions are given, main properties of skew products are described, and it is shown that Theorem A can be reduced to Theorem A'. In Sec. 3, Lemma 1 (Main Lemma), which plays a significant role in the proof of Theorem A', is formulated and proved. In Sec. 4, it is shown that the proof of Theorem A' can be reduced to consideration of two cases: Case (A1) and Case (A2), and Case (A1) is proved. In addition, a scheme of the proof of Theorem A' is briefly outlined at the beginning of Sec. 4. In Sec. 5, two auxiliary lemmas on properties of the skew products under consideration, which are necessary for the proof of Case (A2), are formulated and proved. In Sec. 6, Case (A2) is proved with an exception of Lemma 6. Lemma 6 is proved in Sec. 7, which consists of 4 subsections. In Subsec. 7.1, main notions required for the proof of Lemma 6 are introduced. In Subsec. 7.2, outlines of proofs of Lemmas 8 and 9 playing a key role in the proof of item (6.c) of Lemma 6 are given. In Subsec. 7.3, the proof of item (6.c) is completed; and, finally, in Subsec. 7.4, the remaining items of Lemma 6 are proved.

§2. Dynamical properties of skew products

Let us give main definitions.

By Σ^2 denote the space of all two-sided sequences of 0 and 1 with the metric

$$d_{\Sigma^2}(\omega, \omega') = 1/2^k,$$

where $k \geq 0$ is the minimal integer number such that if $\omega = \dots \beta_{-1} | \beta_0 \beta_1 \dots$ and $\omega' = \dots \beta'_{-1} | \beta'_0 \beta'_1 \dots$, then

$$\beta_{-k-1} \neq \beta'_{-k-1} \quad \text{or} \quad \beta_k \neq \beta'_k,$$

and the sign $|$ means that the next symbol stands at the zero position. The sign $|$ is used further in the paper. Let us recall the definition of the Bernoulli shift $\sigma : \Sigma^2 \mapsto \Sigma^2$:

$$\sigma(\dots \beta_{-1} | \beta_0 \beta_1 \beta_2 \dots) = \dots \beta_{-1} \beta_0 | \beta_1 \beta_2 \dots$$

In the paper [4], the following definitions are introduced:

Definition 1. Fix two diffeomorphisms f_0 and f_1 of the circle S^1 . A step skew product is a mapping $G : \Sigma^2 \times S^1 \mapsto \Sigma^2 \times S^1$ such that

$$G(\omega, \phi) = (\sigma(\omega), f_{\omega_0}(\phi)) \quad \text{for } \omega \in \Sigma^2, \phi \in S^1,$$

where ω_0 is the symbol standing at the zero position of the sequence ω .

Definition 2. Fix a family of diffeomorphisms f_ω of the circle S^1 that is parameterized by two-sided sequences $\omega \in \Sigma^2$. A mild skew product is a mapping $G : \Sigma^2 \times S^1 \mapsto \Sigma^2 \times S^1$ such that

$$G(\omega, \phi) = (\sigma(\omega), f_\omega(\phi)) \quad \text{for } \omega \in \Sigma^2, \phi \in S^1.$$

Let us emphasize that in Definition 1, the choice of the diffeomorphism f_j is completely determined by the symbol ω_0 standing at the zero position of the sequence ω , whereas in Definition 2, the choice depends on the whole sequence ω . By g denote a diffeomorphism of the sphere S^2 that has a standard Smale horseshoe. It is well known that the mapping g has a locally maximal invariant subset Λ homeomorphic to the set Σ^2 , and that the restriction of the mapping g to the set Λ is topologically conjugate with the Bernoulli shift σ (cf., e.g., [8]).

It is well known (cf., e.g., [4]) that the diffeomorphism $g : S^2 \mapsto S^2$ can be considered as a mapping $g : D_0 \cup D_1 \mapsto D'_0 \cup D'_1$, where D_0 and D_1 are disjoint horizontal rectangles, and D'_0 and D'_1 are disjoint vertical rectangles. In the following definition (which is also taken from the paper [4]), we extend a step skew product to the set $(D_0 \cup D_1) \times S^1$:

Definition 3. The smooth realization of a step skew product G is a smooth mapping $F : (D_0 \cup D_1) \times S^1 \mapsto (D'_0 \cup D'_1) \times S^1$ such that

$$F(x, \phi) = (g(x), f_x(\phi)) \quad \text{for } x \in D_0 \cup D_1, \phi \in S^1,$$

$$\text{where } f_x = f_j \quad \text{for } x \in D_j, j \in \{0, 1\},$$

and f_0 and f_1 are the diffeomorphisms from the definition of the step skew product G .

The smooth realization F of a skew product G can be smoothly extended to a diffeomorphism of the manifold $M = S^2 \times S^1$. We denote this extension by F again, and, hereinafter, we understand by a smooth realization precisely a diffeomorphism of the manifold M . It is easily seen that the diffeomorphism F has a locally maximal invariant set such that the dynamics on this set coincides with the dynamics of the initial skew product G .

Let g_0 be the rotation of the circle S^1 by small angle $b < 1/100$. Let g_1 be an orientation preserving diffeomorphism whose non-wandering set consists only of two fixed points: an attractor p and a repeller q . As usual, we consider S^1 as \mathbf{R}/\mathbf{Z} . We assume that the mapping g_1 is the linear expansion with some constant $a > 1$ in the neighborhood of the point $q = 0$ of radius $1/8$ and the linear contraction with the constant $1/a$ in the neighborhood of the point $p = 1/2$ of radius $1/8$. As usual, we denote by id the identity map. In addition, we assume that the diffeomorphisms g_0 and g_1 satisfy the formula

$$\text{dist}_{C^1}(g_j, \text{id}) < \delta_0 \quad \text{for } j \in \{0, 1\},$$

where δ_0 is a sufficiently small number (we will impose restrictions on the size of δ_0 further, in Theorem 2). By G_0 denote the step skew product generated by the diffeomorphisms g_0 and g_1 . G_0 is precisely the skew product discussed in Sec. 1.

The set Σ^2 is called the base, and any set of the form $\omega \times S^1$, where $\omega \in \Sigma^2$, is called a fibre. We denote by $pr : \Sigma^2 \times S^1 \mapsto \Sigma^2$ the natural projection onto the base. We say that the trajectories of points $p_1, p_2 \in \Sigma^2 \times S^1$ lie on different fibres if the trajectories of their projections to the base do not intersect. Any finite sequence of 0 or 1 is called a word.

A definition of a hyperbolic periodic point is given in the book [12]. Let p be a hyperbolic periodic point of a diffeomorphism f of a manifold M . Let us define the sets

$$W^s(p) = \left\{ q \in M \mid \exists r \in O(p, f) : \text{dist} \left(f^k(q), f^k(r) \right) \longrightarrow 0, k \rightarrow +\infty \right\}, \quad (2.1)$$

$$W^u(p) = \left\{ q \in M \mid \exists r \in O(p, f) : \text{dist} \left(f^k(q), f^k(r) \right) \longrightarrow 0, k \rightarrow -\infty \right\}. \quad (2.2)$$

For convenience, we call the sets defined by equalities (2.1) and (2.2) the stable and the unstable manifolds of the point p , respectively. Note that usually this sets are called the stable and unstable manifolds of the trajectory $O(p, f)$. We say that the point p is a point of type (m, n) if

$$\dim W^s(p) = m \quad \text{and} \quad \dim W^u(p) = n.$$

A periodic point $p \in \Sigma^2$ of the Bernoulli shift σ is called a hyperbolic periodic point of type $(1, 1)$. According to this definition, a hyperbolic periodic point of type $(1, 1)$ of the diffeomorphism $g : S^2 \mapsto S^2$ corresponds to any

hyperbolic periodic point $p \in \Sigma^2$. A periodic point $p = (\omega_0, \phi_0) \in \Sigma^2 \times S^1$ is called a hyperbolic periodic point of type (2,1) (of type (1,2), respectively) of a mild skew product G if it is a hyperbolic attracting (repelling) point of the diffeomorphism

$$\bar{G}_{\omega_0, \phi_0} : S^1 \mapsto S^1, \quad \bar{G}_{\omega_0, \phi_0}(\phi) := pr_{S^1} G^{m_p}(\omega_0, \phi) \quad \text{for } \phi \in S^1,$$

where pr_{S^1} is a projection onto S^1 , and m_p is the period of the point p .

We use a result of Ilyashenko and Gorodetski on density of hyperbolic periodic points of different types, which is, in fact, a consequence of Theorem 2 in [5]:

Theorem 1 (Gorodetski, Ilyashenko). *For the diffeomorphisms g_0 and g_1 defined above and any numbers C and α there exist neighborhoods $W_0(g_0)$ and $W_1(g_1)$ (in the C^1 -topology) such that if a mild skew product G (generated by diffeomorphisms f_ω) satisfies the conditions:*

$$f_\omega \in W_{\omega_0} \quad \text{for } \omega \in \Sigma^2 \quad (2.3)$$

(where ω_0 is the symbol standing at the zero position of the sequence ω);

$$L := \max_{\omega \in \Sigma^2} \max_{\phi \in S^1} (||Df_\omega(\phi)||, ||Df_\omega^{-1}(\phi)||) < 2^\alpha; \quad (2.4)$$

$$d_{C^0}(f_\omega, f_{\omega'}) \leq C(d_{\Sigma^2}(\omega, \omega'))^\alpha \quad \text{for } \omega, \omega' \in \Sigma^2, \quad (2.5)$$

where d_{C^0} is the C^0 -metric, then both hyperbolic periodic points of type (2,1) and hyperbolic periodic points of type (1,2) are dense in the set $\Sigma^2 \times S^1$.

Note that if diffeomorphisms f_ω satisfy relation (2.5), then the mild skew product G is called a Holder mild skew product. In fact, Theorem 1 states that hyperbolic periodic points of different types are dense for Holder mild skew products "sufficiently close" to the skew product G_0 .

By F_0 denote the smooth realization of the step skew product G_0 . The following theorem plays an important role in the proof of Theorem A. In fact, the theorem states that any diffeomorphism close to the diffeomorphism F_0 has a local maximal invariant set such that the dynamics on this set coincides with the dynamics of some Holder skew product "close" to the skew product G_0 .

Theorem 2. *Suppose that the diffeomorphisms g_0 and g_1 defined above are sufficiently C^1 -close to the identity diffeomorphism (i.e, the number δ_0 defined above is sufficiently small). Then there exist numbers C and α , and a neighborhood W of the diffeomorphism F_0 in the C^1 -topology such that any diffeomorphism $F \in W$ has a local maximal invariant set Δ , and $F|_\Delta$ is topologically conjugate with a mild skew product G , which satisfies all conditions of Theorem 1 (namely, conditions (2.3), (2.4) and (2.5)).*

Remark 1. It seems that Theorem 2 has not been formulated anywhere, but in fact, its proof is given in the papers [9, 6, 3]. In Theorem 2, a mild skew product G is non-one-to-one assigned to a diffeomorphism F , which is sufficiently close to the identity. In the papers [9, 6, 3], it is proved that this correspondence is continuous. Note that the reasoning of the papers [6, 3] implies that hyperbolic periodic points from Δ of diffeomorphisms from W are assigned to hyperbolic periodic points of the same type of mild skew products.

In order to prove the main result, we need the following notations.

For convenience, we denote by dist both the metric on the manifold $S^2 \times S^1$ and the metric in the space $\Sigma^2 \times S^1$. Let p be a hyperbolic periodic point of a homeomorphism f of a metric space M . The set defined by equality (2.1) (equality (2.2), respectively) is called the stable (respectively, the unstable) manifold of the point p . Choose a mild skew product G from Theorem 2 as a homeomorphism f . Let F be the diffeomorphism of the manifold M that corresponds to the mild skew product G . Let us emphasize, that the sets $W^s(p)$ and $W^u(p)$ are not manifolds in spite of their names. However, the sets $W_F^s(\bar{p})$ and $W_F^u(\bar{p})$, the stable and unstable manifolds of the trajectory of the point $\bar{p} \in S^2 \times S^1$ (which corresponds to the point $p \in \Sigma^2 \times S^1$ in the sense of Theorem 2) with respect to the diffeomorphism F , are the manifolds. Define dimensions of the sets $W^s(p)$ and $W^u(p)$ by the formula

$$\dim W^s(p) := \dim W_F^s(\bar{p}), \quad \dim W^u(p) := \dim W_F^u(\bar{p}).$$

Thus, if p is a hyperbolic periodic point of type (m, n) , then $\dim W^s(p) = m$ and $\dim W^u(p) = n$.

If p is a periodic point of the Bernoulli shift σ , then the numbers $\dim W^s(p)$ and $\dim W^u(p)$ can be defined in the analogous way with the exception that a diffeomorphism F should be changed by the diffeomorphism g defined above. According to this definition, $\dim W^s(p) = \dim W^u(p) = 1$, i.e., the point $p \in \Sigma^2$ is a hyperbolic periodic point of type $(1, 1)$.

Let W be the neighborhood of the mapping F_0 from Theorem 2. In order to prove Theorem 2, it is sufficient to find a number δ' such that $N(\delta', F_0) \subset W$, and any diffeomorphism $F \in N(\delta', F_0)$ does not have OSP. Let δ be an arbitrary number. By Remark 1, if δ' is sufficiently small, then any diffeomorphism $F \in N(\delta', F_0)$ is assigned to some mild skew product G , and the diffeomorphisms f_ω of the mild skew product G (cf., Definition 2) are contained in the neighborhood $N(\delta, g_{\omega_0})$, where the symbol ω_0 was defined in the conditions of Theorem 1. During the proof of Theorem A, we fix a sufficiently small number δ , i.e., we fix the number δ' too, implicitly. By Theorem 2, any diffeomorphism $F \in W$ has the local maximal invariant set Δ . Hence, in order to prove Theorem A, it is enough to establish just that the restriction $F|_\Delta$ does

not have OSP. Since OSP is preserved under conjugacy, Theorem A is reduced to the following theorem:

Theorem A'. *Let G be a mild skew product correspondent (in the sense of Remark 1) to some diffeomorphism F of the manifold $M = S^2 \times S^1$. In addition, suppose that the skew product G satisfies the conditions of Theorem 1, and the neighborhoods $W_0(g_0)$ and $W_1(g_1)$ are sufficiently small. Then G does not have OSP.*

§3. Main lemma.

The following lemma is a main ingredient of our constructions. It gives a sufficient condition for a pseudotrajectory to satisfy the following: any exact trajectory that orbitally shadows the pseudotrajectory is contained in the stable (or the unstable) manifold of a hyperbolic fixed point.

Lemma 1. *Let M be a closed smooth manifold with metric dist , f be a diffeomorphism of the manifold M , p be a hyperbolic periodic point, and $q^1 \in W^s(p)$. Fix numbers $R > 0$ and $0 < \epsilon_0 < R/2$. There exists a number $0 < \epsilon < \epsilon_0$ such that if a sequence $\xi = \{x_k\}$ satisfies the following relations:*

$$x_k = q_k^1 = f^k(q^1) \quad \text{for } k \geq 1, \quad x_k \notin N(\epsilon_0, O(p, f)) \quad \text{for } k < 1, \quad (3.1)$$

then for any point q^2 which satisfies the inclusions

$$\xi \subset N(\epsilon, O(q^2, f)) \quad \text{and} \quad O(q^2, f) \subset N(\epsilon, \xi) \quad (3.2)$$

and the inequality $\text{dist}(q_1^1, q_1^2) < \epsilon$ the following holds:

$$q^2 \in W^s(p), \quad (3.3)$$

$$\text{dist}(q_k^1, q_k^2) \leq R \quad \text{for } k \geq 1. \quad (3.4)$$

In fact, the lemma states the following: Let p be a hyperbolic periodic point. Then, if a pseudotrajectory $\xi = \{x_k\}$ of a certain type is "close" to the trajectory $O(p, f)$ for all sufficiently large positive k and is "far" from the trajectory $O(p, f)$ for all negative k with sufficiently large absolute values, then any exact trajectory that orbitally shadows the pseudotrajectory ξ is contained in the stable manifold of the point p .

Corollary. *Under the conditions of Lemma 1, suppose that $q^1 \in W^u(p)$. There exists a number $0 < \epsilon < \epsilon_0$ such that if a sequence $\xi = \{x_k\}$ satisfies the relations*

$$x_k = q_k^1 = f^k(q^1) \quad \text{for } k \leq 1, \quad x_k \notin N(\epsilon_0, O(p, f)) \quad \text{for } k > 1, \quad (3.5)$$

then for any point q^2 which satisfies the inequality $\text{dist}(q_1^1, q_1^2) < \epsilon$ and inclusions (3.2) the following holds:

$$q^2 \in W^u(p), \quad (3.6)$$

$$\text{dist}(q_k^1, q_k^2) \leq R \quad \text{for } k \leq 1. \quad (3.7)$$

Proof of Lemma 1. At first, we need to choose a sufficiently small ϵ . We do it in several steps.

Choose a number $\epsilon_1 < \epsilon_0$ such that

- if $O(p, f) = \{p_0, \dots, p_{m_p-1}\}$, then the neighborhoods

$$N(\epsilon_1, p_0), N(\epsilon_1, p_1), \dots, N(\epsilon_1, p_{m_p-1})$$

are disjoint;

- if a positive semitrajectory $O_+(x, f)$ is contained in the set $N(\epsilon_1, O(p, f))$, then $x \in W^s(p)$;
- $(N(\epsilon_1, O(p, f)) \cup f(N(\epsilon_1, O(p, f)))) \cap N(\epsilon_1, x_k) = \emptyset$ for $k \leq 1$;
- there are no points of the sequence ξ on the boundary of the set $N(\epsilon_1, O(p, f))$.

Let $n > 1$ be the minimal number such that the points x_k are contained in the neighborhood $N(\epsilon_1, O(p, f))$ for $k \geq n$. Choose a number $\epsilon_2 < \epsilon_1$ such that

- the following neighborhoods are disjoint:

$$N(\epsilon_2, O(p, f)) \quad \text{and} \quad N(\epsilon_2, x_k) \quad \text{for } 1 \leq k \leq n;$$

- if there exist numbers $1 \leq k_1 \leq n$ and $1 \leq k_2$ such that

$$x_{k_1} = q_{k_1}^1 \notin N(\epsilon_1, O(p, f)) \quad \text{and} \quad x_{k_2} = q_{k_2}^1 \in N(\epsilon_1, O(p, f)),$$

then $N(\epsilon_1, O(p, f)) \cap N(\epsilon_2, x_{k_1}) = \emptyset$ and $N(\epsilon_2, x_{k_2}) \subset N(\epsilon_1, O(p, f))$.

Let $m \geq n$ be the minimal number such that the point x_m is contained in the set $N(\epsilon_2/3, O(p, f))$. Choose a number $\epsilon_3 < \epsilon_2/3$ such that the neighborhoods $N(\epsilon_3, x_k)$ are disjoint for $1 \leq k \leq m$. Choose a number $\epsilon < \epsilon_3$ such that

$$f^{j-k}(N(\epsilon, x_k)) \subset N(\epsilon_3, x_j) \quad \text{for } j, k \in \{1, \dots, m\}.$$

Let us show that the number ϵ chosen above has the desired properties. Suppose that relation (3.2) holds for a point q^2 and for the sequence ξ .

Since $q_1^2 \in N(\epsilon, x_1)$, the choice of ϵ implies that

$$q_k^2 \in N(\epsilon_3, x_k) \quad \text{for } 1 \leq k \leq m. \quad (3.8)$$

Thus, by inclusion (3.8) and our notations, the point q_2^m is contained in the set $N(\epsilon_2, O(p, f))$. In order to obtain inclusion (3.3), it is sufficient to prove the following inclusion

$$q_k^2 \in N(\epsilon_1, O(p, f)) \quad \text{for } k \geq m. \quad (3.9)$$

Suppose that inclusion (3.9) does not hold, i.e., there exists the minimal number $r > m$ such that

$$q_r^2 \notin N(\epsilon_1, O(p, f)). \quad (3.10)$$

From relation (3.2) and the choice of ϵ_2 and ϵ_3 it follows that there exists a number $k < n$ such that the point q_r^2 is contained in the set $N(\epsilon, x_k)$. Two cases are possible:

Case 1: $k \leq 1$. By the choice of r , $q_{r-1}^2 \in N(\epsilon_1, O(p, f))$. But then, by relation (3.10),

$$f(N(\epsilon_1, O(p, f))) \cap N(\epsilon_1, x_k) \neq \emptyset \quad \text{for } k \leq 1,$$

what contradicts to the choice of ϵ .

Case 2: $1 < k < n$. There exists the maximal number $1 \leq k' < k$ such that the point $x_{k'}$ does not belong to the set $N(\epsilon_1, O(p, f))$. But then, by the choice of ϵ ,

$$q_{r-t}^2 \in N(\epsilon_3, x_{k-t}) \quad \text{for } 0 \leq t \leq k - k',$$

and, hence, by the choice of ϵ_2 ,

$$q_{r-(k-k')}^2 \notin N(\epsilon_1, O(p, f)) \quad \text{and} \quad q_{r-t}^2 \notin N(\epsilon_2, O(p, f)) \quad \text{for } 0 \leq t \leq k - k'.$$

Thus, $r - (k - k') > m$ (since, otherwise, one could find a number $1 \leq t \leq k - k'$ such that the point $q_m^2 = q_{r-t}^2$ belongs to the set $N(\epsilon_2, O(p, f))$). Hence, the number r is not minimal, and we get the desired contradiction.

We proved inclusion (3.9) and, hence, inclusion (3.3) too. By relations (3.8) and (3.9), and since $q_k^1 \in N(\epsilon_1, O(p, f))$ for $k \geq m$, inequality (3.4) holds for the points q^1 and q^2 . Lemma 1 is proved. \square

§4. Reduction of the proof of Theorem A' to two cases, the proof in Case (A1)

Two following cases are possible:

- (A1) There exist two hyperbolic periodic, lying on different fibres, points r_1 and r_2 with one-dimensional unstable and stable manifolds, respectively, such that this manifolds intersect.
- (A2) For any hyperbolic periodic, lying on different fibres, points r_1 and r_2 with one-dimensional unstable and stable manifolds, respectively, this manifolds do not intersect.

Now we can give a more detailed outline of the proof scheme in the two cases under consideration.

In Case (A1), we construct the pseudotrajectory in the following way: It includes a part of an exact heteroclinic trajectory from r_1 to r_2 , then it "leaps" (sufficiently close to the point r_2) to a trajectory that lies on the fibre of the

point r_2 and "goes away" from the point r_2 by its unstable manifold. Let us emphasize that Main Lemma can be applied not only to the pseudotrajectory, but also to its projection to the base (the Smale horseshoe). To get a contradiction, suppose that the constructed pseudotrajectory is orbitally shadowed by an exact trajectory. Next, we apply Main Lemma to the projection of the trajectory to the base and see that this trajectory lies on the fibre of some trajectory "going to" the point r_2 . On the other hand, since the pseudotrajectory "goes away" from the point r_1 , the exact trajectory should be contained in the unstable manifold of the point r_1 . In addition, since the Bernoulli shift is expansive, the projection to the base of the heteroclinic trajectory used in our construction should coincide with the projection of the exact shadowing trajectory. Finally, since the projection of a local unstable manifold is one-to-one in a small neighborhood of the point r_1 , and we know the point in the base, the exact trajectory should precisely coincide with the heteroclinic trajectory used in our construction. We see the contradiction: the "final" part of the pseudotrajectory (near the point r_2) is not shadowed. The phase portrait of a mild skew product in Case (A1) is depicted in Fig. 1.

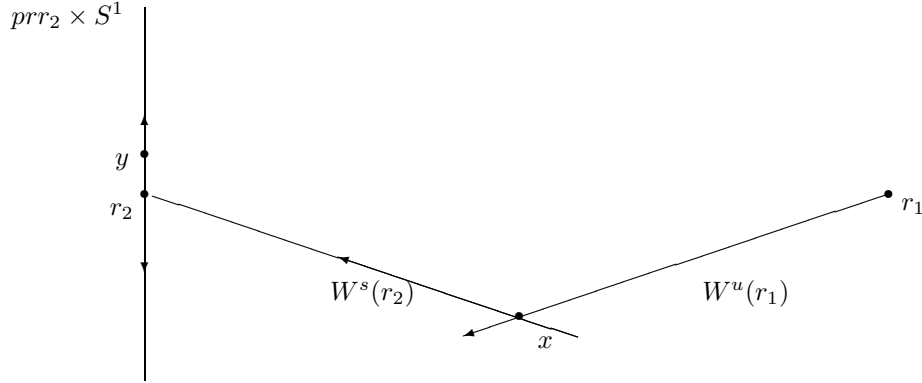


Fig. 1

In Case (A2), using the technique of mild skew products, we construct a pseudotrajectory that "goes from" and "goes to" hyperbolic periodic points with one-dimensional unstable and stable manifolds, respectively, in the initial and final phases, respectively, and does not approach this points in the intermediate phase. Then, by Main Lemma, if an exact trajectory orbitally shadows the pseudotrajectory, then it should be a heteroclinic trajectory connecting this points, and we get the desired contradiction.

Case (A1) is proved in this section, and Case (A2) is considered in the remaining sections.

Suppose that a mild skew product G satisfies the conditions of Case (A1). Choose points $x \in W^u(r_1) \cap W^s(r_2)$ and $y \in (prr_2 \times S^1) \cap W^u(r_2)$ (cf., Fig. 1). At first, we need to choose a sufficiently small number ε . We do it in several steps.

Choose a number $\varepsilon_0 > 0$ such that

- 1) the statement of Main Lemma corollary holds for the mapping G , some point of the trajectory $O(x, G)$ and the point r_1 ;
- 2) the statement of Main Lemma holds for the Bernoulli shift σ , some point of the trajectory $O(prx, \sigma)$ and the point prr_2 (in this two items the number R is assumed to be a sufficiently small, but preliminary fixed number);
- 3) the point y does not belong to the set $N(2\varepsilon_0, O(x, G))$;
- 4) the restriction of the projection pr onto a local unstable manifold $W_{\varepsilon_0}^u(r_1)$ is a one-to-one mapping.

Let us explain items 1) and 2). Strictly speaking, both Main Lemma and its corollary can not be applied directly in our case, since the spaces Σ^2 and $\Sigma^2 \times S^1$ are not manifolds, and the above-mentioned statements were proved only for the manifolds. However, in item 1), we can consider the diffeomorphism F of the manifold M that corresponds to the mild skew product G (in the sense of Remark 1), and points \bar{x} and \bar{r}_1 which are the analogs of the points x and r_1 for the diffeomorphism F . There exist a number ϵ_0 and a point $x_0 \in O(x, G)$ that satisfy the analog of condition (3.5). Choose a point $q^2 \in \Sigma^2 \times S^1$ that satisfies the analog of relation (3.2) for some small ϵ and the inequality $\text{dist}(G(q^2), G(x_0)) < \epsilon$. By Theorem 2, there exists a homeomorphism h_F conjugating the restriction $F|_\Delta$ (where the set Δ was defined in the conditions of Theorem 2) and the mild skew product G . Since the homeomorphisms h_F and h_F^{-1} are homeomorphisms of compact metric spaces, they are uniformly continuous. Hence, the analog of condition (3.5) holds for some number $\bar{\epsilon}_0$ and point $\bar{x}_0 = h_F^{-1}(x_0)$. Moreover, all conditions of Corollary are satisfied for the point $\bar{q}^2 = h_F^{-1}(q^2)$, an analog of the point q^2 , the point \bar{x}_0 , sufficiently small number $\bar{\epsilon}$, and the mapping F . Thus, Corollary can be applied to the diffeomorphism F , i.e., the analogs of relations (3.6) and (3.7) hold for sufficiently small numbers \bar{R} and $\bar{\epsilon}$, and the points \bar{x}_0 and \bar{q}^2 . Hence, by the uniform continuity of the homeomorphism h_F , the analogs of relations (3.6) and (3.7) hold for sufficiently small numbers R and ϵ , and the points x_0 and q^2 .

Hence, the statement of Corollary holds for the mapping G . In item 2), we can consider the diffeomorphism $g : S^2 \mapsto S^2$ (which was fixed above)

having the Smale horseshoe and apply similar reasoning. Thus, the statement of Lemma 1 can be applied to the Bernoulli shift σ too.

Let us explain item 4). Consider the diffeomorphism $F : M \mapsto M$ corresponding to the mild skew product G . By Theorem 2, the diffeomorphism F has the local maximal invariant set Δ homeomorphic to $\Sigma^2 \times S^1$ such that $F|_\Delta$ and G are topologically conjugate. The sets homeomorphic to S^1 and corresponding to the fibres of the mild skew product G are called the center fibres. By Remark 1, any periodic point r_1 of the mild skew product G is assigned to some hyperbolic periodic point \bar{r}_1 of the diffeomorphism F . Since r_1 is a point of type $(2, 1)$, a local unstable manifold of the point \bar{r}_1 with respect to the diffeomorphism F , i.e., a set $W_{\text{loc}, F}^u(\bar{r}_1)$, is just a finite union of "intervals". The angle between the unstable space of the diffeomorphism F at the point \bar{r}_1 and the corresponding central fibre is not equal to zero. It is shown in the papers [9, 6, 7] that the central fibres of the diffeomorphism F are C^1 -close to the corresponding fibres of the diffeomorphism $g \times \text{id}_{S^1}$, i.e., to the circles. Hence, the intersection of the set $W_{\text{loc}, F}^u(\bar{r}_1)$ and any central fibre consists of no more than one point. Consequently, the intersection of the set $W_{\text{loc}, G}^u(r_1)$ and any fibre also consists of no more than one point. Thus, the projection to the base is a one-to-one mapping on the set $W_{\text{loc}, G}^u(r_1)$.

We need the following two lemmas:

Lemma 2. *There exists a number R such that if $\sigma : \Sigma^2 \mapsto \Sigma^2$ is the Bernoulli shift, and points $q^1, q^2 \in \Sigma^2$ satisfy the inequality*

$$d_{\Sigma^2}(q_k^1, q_k^2) \leq R \quad \text{for } k \in \mathbf{Z},$$

then $q^1 = q^2$ (recall that $q_k^j = \sigma^k(q^j)$).

In fact, Lemma 2 means that the Bernoulli shift is expansive. The proof of this fact is given, e.g., in the book [8].

Lemma 3. *Under the conditions of Case (A1), let R be the number given by Lemma 2. There exists an $\varepsilon < \varepsilon_0$ such that if the relations*

$$d_{\Sigma^2}(q_1^1, q_1^2) < \varepsilon,$$

$$O(q^1, \sigma) \subset N(\varepsilon, O(q^2, \sigma)) \quad \text{and} \quad O(q^2, \sigma) \subset N(\varepsilon, O(q^1, \sigma))$$

hold for two points $q^1 = prx$ and q^2 from Σ^2 such that $q^1, q^2 \in W^u(prr_1) \cap W^s(prr_2)$, then $\text{dist}(q_k^1, q_k^2) \leq R$ for $k \in \mathbf{Z}$.

In other words, Lemma 3 means that, under the conditions of Case (A1), if two heteroclinic trajectories which "go from" prr_1 "to" prr_2 are orbitally close, then they are pointwise close.

Proof. By the choice of ε_0 , the statement of Lemma 1 corollary holds for the point prr_1 , the mapping σ and some point $q_{k_1-1}^1 = \sigma^{k_1-1}(q^1) \in O(prx, \sigma)$, and the statement of Lemma 1 holds for the point prr_2 , the mapping σ and some point $q_{k_2-1}^2 \in O(prx, \sigma)$ (with R given by Lemma 2). Hence, if $d_{\Sigma^2}(q_{k_1}^1, q_{k_1}^2) < \varepsilon_0$, then, by inequalities (3.4),

$$d_{\Sigma^2}(q_k^1, q_k^2) \leq R \quad \text{for } k \leq k_1.$$

Similarly, if $d_{\Sigma^2}(q_{k_2}^1, q_{k_2}^2) < \varepsilon_0$, then

$$d_{\Sigma^2}(q_k^1, q_k^2) \leq R \quad \text{for } k \geq k_2.$$

Choose a number $\varepsilon < \varepsilon_0$ such that the inequalities $d_{\Sigma^2}(q_1^1, q_1^2) < \varepsilon$ imply the inequalities

$$d_{\Sigma^2}(q_k^1, q_k^2) < \varepsilon_0 \quad \text{for } k \text{ between } 1 \text{ and } k_1,$$

$$d_{\Sigma^2}(q_k^1, q_k^2) < \varepsilon_0 \quad \text{for } k \text{ between } 1 \text{ and } k_2.$$

The number ε has the desired properties, i.e., the statement of Lemma 3 holds for this number. \square

Let ε be the number whose existence was proved in Lemma 3. Choose an arbitrary number $d < \varepsilon$. Now we construct the pseudotrajectory discussed at the beginning of Sec. 4. Choose numbers k_1 and k_2 such that

$$x_{k_1+1} \in N(d/2, O(r_2, G)) \quad \text{and} \quad y_{k_2} \in N(d/2, O(r_2, G)).$$

We construct the d -pseudotrajectory $\xi = \{\xi_k\}$ in the following way (cf. Fig. 1):

$$\xi_k = x_k \text{ for } k \leq k_1, \quad \xi_k = y_{k-k_1+k_2-1} \text{ for } k > k_1.$$

Suppose that the mild skew product G has OSP. Then there exists the point q such that relation (1.1) holds for it and the pseudotrajectory ξ .

By the choice of ε , the corollary of Lemma 1 holds for the point r_1 and the pseudotrajectory ξ . Hence,

$$q \in W^u(r_1). \quad (4.1)$$

Consider the sequence $pr\xi$. The points of the sequence $pr\xi$ coincide with the corresponding points of the trajectory $O(prr_2, \sigma)$ before the intersection with the $d/2$ -small neighborhood of the point prr_2 . Therefore, by relation (1.1),

$$O(prx, \sigma) \subset N(\varepsilon, O(prq, \sigma)) \quad \text{and} \quad O(prq, \sigma) \subset N(\varepsilon, O(prx, \sigma)) \quad (4.2)$$

for a sufficiently small d . Thus, relation (4.2) and the analog of relation (3.1) hold for the trajectory of the point prx . Hence, by the choice of ε and by relation (4.1),

$$prq \in W^u(prr_1) \cap W^s(prr_2).$$

All conditions of Lemma 3 hold for the points prx and prq (of course, we can assume that $\text{dist}(x_1, q_1) < \varepsilon$). Hence,

$$d_{\Sigma^2}(\sigma^k(prx), \sigma^k(prq)) \leq R \quad \text{for } k \in \mathbf{Z}.$$

By Lemma 2,

$$prq_k = prx_k \quad \text{for } k \in \mathbf{Z}.$$

Since $x, q \in W^u(r_1)$, one can find a number K such that the points x_K and q_K belong to a local unstable manifold of the point r_1 that can be projected to the base injectively (cf., the choice of ε , item 4)). Hence, the equality $prq_K = prx_K$ implies the equality $q_K = x_K$, and the latter one implies the equality $q = x$. Consequently, by relation (1.1), the following inclusions hold:

$$O(x, G) \subset N(\varepsilon, \xi) \quad \text{and} \quad \xi \subset N(\varepsilon, O(x, G)),$$

what contradicts to the construction of the sequence ξ . The derived contradiction means that our assumptions are wrong, and $G \notin \text{OSP}$ in Case (A1).

§5. Start of the proof in Case (A2): auxiliary lemmas

Two auxiliary lemmas on properties of skew products necessary for the proof in Case (A2) will be formulated and proved in this section. Let us introduce corresponding notations.

Consider the step skew product G_0 generated by the diffeomorphisms g_0 and g_1 . By Theorem 1, the step skew product G_0 has an infinite number of hyperbolic periodic points of type (1,2) lying on different fibres and an infinite number of hyperbolic periodic points of type (2,1) lying on different fibres. Note that any infinite set of lying on different fibres periodic points in the space $\Sigma^2 \times S^1$ contains points of arbitrary large periods. Choose four hyperbolic periodic, lying on different fibres, points of the step skew product G_0 : points p_1 and p_3 of type (2,1), and points p_2 and p_4 of type (1,2).

Recall that any finite sequence of zeros or ones is called a word. The sequence prp_j , where $j \in \{1, \dots, 4\}$, is periodic, i.e., some word ω_j of length T_j is periodically repeated in it. We can assume that the word ω_j is the word of minimal length, i.e., the number T_j is the main period of the point prp_j with respect to the Bernoulli shift σ . Without loss of generality, we assume that $T_j > 2$ for $j \in \{1, \dots, 4\}$, i.e., the word ω_j contains both 0 and 1. In addition, without loss of generality, we assume that

$$T_3 \leq \min(T_1, T_2) \quad \text{and} \quad T_4 \leq \min(T_1, T_2). \quad (5.1)$$

By definition, put

$$T = \max(T_1, T_2). \quad (5.2)$$

We can assume that the neighborhoods W_0 and W_1 from Theorem A' are so small that the points p_j are preserved for any mild skew product (which is, in fact, a perturbation of the step skew product G_0) from Theorem A'. It means that the analogs of this points have the same periods and types. In particular, the number T does not depend on the choice of a mild skew product G . Let δ be the maximal radius of the neighborhoods W_0 and W_1 . We can assume that it is an arbitrary small (and dependent on T) but fixed number. Main restrictions on the size of δ will be imposed further, in Sec. 7.

We denote by the same symbols p_j the hyperbolic periodic points of the mild skew product G corresponding to the points p_j of the step skew product G_0 . As it was noted above, the periods T_j and the types of the points p_j have not changed. Suppose that, as before, ω_j ($j \in \{1, \dots, 4\}$) are periodically repeating words of the sequences prp_j .

Let us define the cylinder neighborhoods U_j of the points p_j ($j \in \{1, 2\}$) by the formula

$$U_j := \{\omega = \dots \omega_j \omega_j | \omega_j \omega_j \dots\} \times S^1.$$

In the previous formula, the dots denote arbitrary symbols, and the meaning of the sign $|$ was explained above (cf., Sec. 2). The word ω_j is repeated four times: two times before the zero position and two times after it. Define the cylinder neighborhoods of the trajectories $O(p_j, G)$ ($j \in \{1, 2\}$) by the formula

$$V_j := U_j^0 \cup U_j^1 \cup \dots \cup U_j^{T_j-1},$$

where the set U_j^k ($0 \leq k \leq T_j - 1$) is defined similarly with the set U_j , only the word ω_j is changed by the word $\sigma^k(\omega_j)$, i.e., the corresponding cyclic permutation of the word ω_j .

Lemma 4. *Under our conditions,*

$$O(p_j, G) \cap V_t = \emptyset \quad \text{for } j \in \{3, 4\}, \quad t \in \{1, 2\}, \quad (5.3)$$

i.e., the trajectories of the points p_3 and p_4 do not intersect the cylinder neighborhoods V_1 and V_2 of the trajectories $O(p_1, G)$ and $O(p_2, G)$.

Proof. Without loss of generality, we prove relation (5.3) for the point p_3 and the set U_1 . To get a contradiction, assume that this relation does not hold. It means that there exists a number K such that the word $\omega_1 \omega_1 \omega_1 \omega_1$ takes the positions from $K - 2T_1$ to $K + 2T_1 - 1$ in the sequence prp_3 . In addition, by relation (5.1), the word ω_1 is longer than the word ω_3 . Consider the word ω_1 starting from the K -position of the sequence prp_3 . The first T_3 symbols of this word are a cyclic permutation of the word ω_3 . Denote this permutation by $\bar{\omega}_3$. Hence, the word ω_1 is covered by m -times repeated words $\bar{\omega}_3$ plus an "addition": r first symbols from $\bar{\omega}_3$ ($0 \leq r < T_3$).

However, the second word ω_1 (the one that starts from the $(K+T_1)$ -position of the sequence prp_3 , cf. Fig. 2) is also covered by words $\bar{\omega}_3$. On the one hand, it should start from $\bar{\omega}_3$ (since the first and the second words ω_1 coincide); on the other hand, it should start from last $T_3 - r$ symbols of $\bar{\omega}_3$ (cf. Fig. 2). It means that if we swap first r symbols and last $T_3 - r$ symbols in the word $\bar{\omega}_3$, then the word $\bar{\omega}_3$ will not change, i.e., $\sigma^r(prp_3) = prp_3$. Hence, $r = 0$, but then the word ω_1 is the m -times repeated word $\bar{\omega}_3$. Thus, the trajectories of the points p_1 and p_3 intersect, and we get a contradiction with the choice of the points p_1 and p_3 . Relation (5.3) is proved. \square

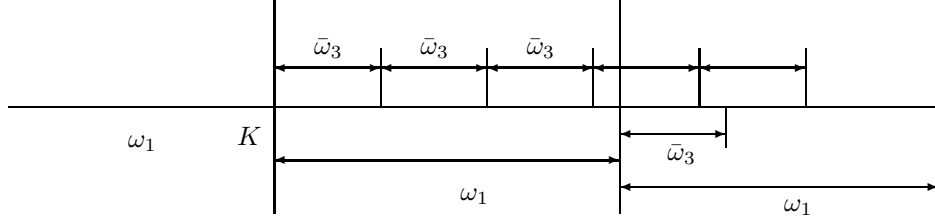


Fig. 2

Lemma 5. Suppose that $m > 4T$ and a sequence $\beta \in \Sigma^2$ is such that a word

$$\omega = \alpha_1 \omega_3 \dots \omega_3 \alpha_2$$

is repeated in it periodically, where the word ω_3 is repeated precisely m times in the formula, and the words α_1 and α_2 (whose length is more than $4T$) can not contain less than T zeros in a row (however, it is allowed not to contain any zeros at all); then

$$O(\beta, \sigma) \cap prV_t = \emptyset \quad \text{for } t \in \{1, 2\}. \quad (5.4)$$

In other words, the trajectory of the sequence β with respect to the Bernoulli shift σ that includes the word ω periodically does not intersect the sets V_1 and V_2 .

Proof. Without loss of generality, we prove that relation (5.4) holds for the point β and the set prU_1 . To get a contradiction, suppose the contrary. It means that there exists a number K such that the word $\omega_1 \omega_1 \omega_1 \omega_1$ takes the positions from $K - 2T_1$ to $K + 2T_1 - 1$ in the sequence β . Two cases are possible:

a) The K -position is "included" in the word α_1 (in the case when it is included in the word α_2 , we can apply the same reasoning). By construction, the word ω_1 contains both zeros and ones. Hence, the word ω_1 can not contain

more than $T_1 - 1$ zeros. This fact and the fact that the word $\omega_1\omega_1$ is contained in the word α_1 contradict to the properties of α_1 .

b) The K -position is "included" in the word $\omega_3 \dots \omega_3$. Since the word ω_3 is repeated m times and $m > 4T$, the word $\omega_1\omega_1$ is covered by a finite number of words ω_3 . Further reasoning for deriving the contradiction is similar with the proof of Lemma 4.

We got the contradiction in both possible cases. Hence, our assumptions are wrong. Lemma 5 is proved. \square

§6. Reduction of the proof in Case (A2) to Lemma 6

Recall that a sketch of the proof in Case (A2) was outlined at the beginning of Sec. 4. In this section we give the proof in Case (A2) with an exception of one lemma.

Choose a mild skew product G from Theorem A' and suppose that it satisfies the conditions of Case (A2) (cf. the definition at the begining of Sec. 4). The following lemma plays a key role in the proof of Case (A2). In fact, it states that it is possible to construct "as precise as we want" pseudotrajectories with the required properties (they should "go from" the point p_1 and "go to" the point p_2 , and their "intermediate part" should be "separated" from the trajectories $O(p_1, G)$ and $O(p_3, G)$).

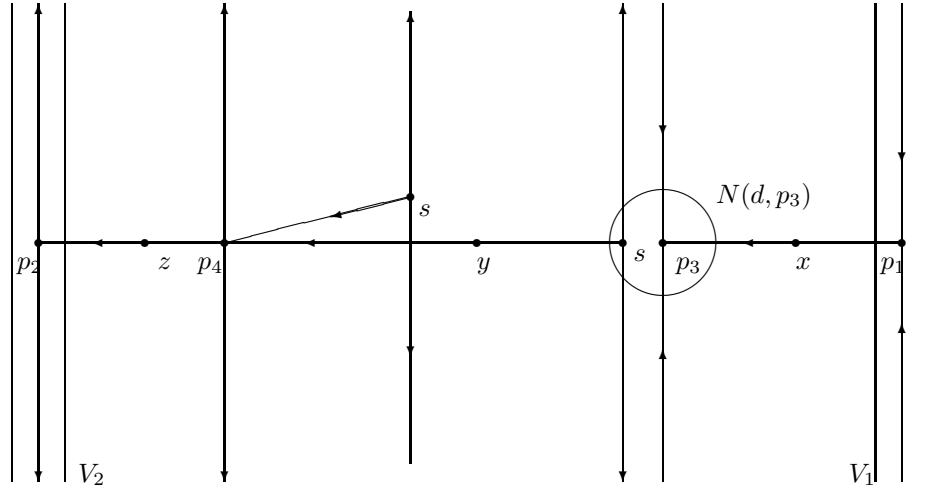


Fig. 3

Lemma 6. *Under our conditions, if δ is sufficiently small, then*

- (6.a) *the one-dimensional unstable manifold of the point p_1 and the two-dimensional stable manifold of the point p_3 intersect;*

- (6.b) *the two-dimensional unstable manifold of the point p_4 and the one-dimensional stable manifold of the point p_2 intersect;*
- (6.c) *given any d , there exists a hyperbolic periodic point s such that*
 - *$s \in N(d, p_3)$, and the unstable manifold of the point s is one-dimensional,*
 - *the trajectory $O(s, G)$ does not intersect the sets V_1 and V_2 ;*
- (6.d) *there exists a point $y \in W^u(s) \cap W^s(p_4)$ whose trajectory $O(s, G)$ does not intersect the sets V_1 and V_2 .*

Lemma 6 will be proved in Sec. 7. Choose points $x \in W^u(p_1) \cap W^s(p_3)$ and $z \in W^u(p_4) \cap W^s(p_2)$. The phase portrait of the mild skew product G is depicted in Fig. 3. For convenience, the symbol s denotes all points of the trajectory $O(s, G)$ from Lemma 6. Let us show how to finish the proof for Case (A2) using Lemma 6. At first, we need to choose a sufficiently small number ε . We do it in several steps.

The points x and z can be chosen so close to the points p_1 and p_2 that there exists a number ε_0 such that

$$\begin{aligned} N(\varepsilon_0, O_-(x, G)) &\subset V_1, \quad N(\varepsilon_0, O_+(z, G)) \subset V_2; \\ N(\varepsilon_0, O_-(x, G)) \cap N(\varepsilon_0, O(z, G) \cup O(y, G)) &= \emptyset; \\ N(\varepsilon_0, O_+(z, G)) \cap N(\varepsilon_0, O(x, G) \cup O(y, G)) &= \emptyset, \end{aligned}$$

where y is an arbitrary point from Lemma 6. There exists a number $\varepsilon < \varepsilon_0/3$ such that

- the statement of Main Lemma holds for the mapping G , the point z and the point p_2 ;
- the statement of Main Lemma corollary holds for the mapping G , the point x and the point p_1 .

It was shown in Sec. 4 (cf., the choice of ε_0), that both Main Lemma and its corollary can be applied to the mild skew products.

Now, when ε is chosen, we can construct the desired pseudotrajectory. Choose an arbitrary number $d < \varepsilon$. Suppose that s is a point corresponding to the number $d/3$, and y is a point from item (6.d) corresponding to the point s whose existence is proved in Lemma 6. There exist numbers k_1, k_2, k_3 and k_4 such that

$$\begin{aligned} x_{k_1+1} &\in N(d/3, p_3), \quad y_{k_2} \in N(d/3, s), \\ y_{k_3+1} &\in N(d/2, p_4), \quad z_{k_4} \in N(d/2, p_4). \end{aligned}$$

Let us construct the d -pseudotrajectory $\xi = \{\xi_k\}$ in the following way:

$$\begin{aligned} \xi_k &= x_k \text{ for } k \leq k_1, \quad \xi_k = y_{k-k_1-1+k_2} \text{ for } k_1 < k \leq k_1 + 1 + k_3 - k_2, \\ \xi_k &= z_{k-k_1-2-k_3+k_2+k_4} \text{ for } k > k_1 + 1 + k_3 - k_2. \end{aligned}$$

Suppose that the mild skew product G has OSP, i.e., there exists a point q such that relation (1.1) holds for the point q and the pseudotrajectory ξ .

By the choice of ε , the statement of Lemma 1 holds for the constructed pseudotrajectory ξ and the point p_2 . Hence,

$$p \in W^s(p_2). \quad (6.1)$$

By similar reasons, the statement of Lemma 1 corollary holds for the pseudotrajectory ξ and the point p_1 . Hence,

$$p \in W^u(p_1). \quad (6.2)$$

But existence of a point p that satisfies both inclusions (6.1) and (6.2) contradicts the conditions of Case (A2). The derived contradiction means that in Case (A2) $G \notin \text{OSP}$.

Thus, in both possible cases we concluded that $G \notin \text{OSP}$. In order to finish the proof of Theorem A', it remains to prove only Lemma 6.

§7. Proof of Lemma 6

The proof of Lemma 6 is based on the proofs of certain lemmas from [5], in a great extent.

7.1. Item (6.c): main notations. We start from the proof of item (6.c). By Theorem 1, there exists a hyperbolic periodic point s that satisfies all conditions of item (6.c) except, perhaps, the last one:

$$O(s, G) \cap (V_1 \cup V_2) = \emptyset. \quad (7.1)$$

In fact, we repeat the major part of proof of Theorem 1 (which was formulated in Sec. 1) from the paper [5], but we need to check that, in addition to other properties, the point s can be required to satisfy relation (7.1). The idea of the proof is to construct the point s in such a way that it would satisfy the conditions of Lemma 5. Then, relation (7.1) holds, by Lemma 5.

We assume that the sets $W_0(g_0)$ and $W_1(g_1)$ are the balls of radius δ . Let us introduce the following notations:

$$\bar{f}_m[\omega] = f_{\sigma^{m-1}(\omega)} \circ \dots \circ f_{\sigma(\omega)} \circ f_\omega,$$

$$\bar{f}_{-m}[\omega] = f_{\sigma^{-m}(\omega)}^{-1} \circ \dots \circ f_{\sigma^{-1}(\omega)}^{-1},$$

$$\bar{f}_0[\omega] = \text{id}.$$

We need the following lemma:

Lemma 7 (Lemma on the errors). *There exists a number K independent on the choice of δ such that if the inequality*

$$d_{\Sigma^2}(\omega, \omega') \leq 2^{-m}$$

holds for a number $m \in \mathbf{N}$ and points $\omega, \omega' \in \Sigma^2$, then

$$d_{C^0}(\bar{f}_{\pm m}[\omega], \bar{f}_{\pm m}[\omega']) \leq \gamma := K\delta^\beta,$$

where $\beta := 1 - \frac{\ln L}{\ln 2^\alpha}$, and numbers L and α were defined in the conditions of Theorem 1.

Remark 2. The proof of Lemma 7 is an almost word-by-word repetition of the proof of Lemma 3.1 from the paper [5]: it is necessary to make some trivial changes concerning the transition to the C^1 -topology. We do not give it here.

We need notations from the paper [5]. Consider the word $\bar{\beta} = \beta_{-m} \dots \beta_{m-1}$. By definition, put

$$C_{\bar{\beta}} = \{\omega = \{\alpha_k\}_{k \in \mathbf{Z}} \in \Sigma^2 \mid \alpha_k = \beta_k \text{ for } -m \leq k \leq m-1\}.$$

The set $C_{\bar{\beta}}$ is a cylinder neighborhood in the set Σ^2 . By definition, put

$$V_{\pm}[\bar{\beta}](\phi) = \{\bar{f}_{\pm m}[\omega](\phi) \mid \omega \in C_{\bar{\beta}}\}.$$

Put $\Gamma_m = C_{\bar{\beta}}$ for a fixed word $\bar{\beta} = \beta_{-m} \dots \beta_0 \dots \beta_{m-1}$. Define the sets $V_m(\phi)$ and $V_{-m}(\phi)$ for $\phi \in S^1$ by the relations

$$V_{\pm m}(\phi) = \{\bar{f}_{\pm m}[\omega](\phi) \mid \omega \in \Gamma_m\}.$$

Note that, by definition, $V_{\pm}[\bar{\beta}](\phi) = V_{\pm m}(\phi)$.

By Lemma 7,

$$\text{diam} V_{\pm m}(\phi) \leq \gamma$$

not depending on the choice of a point ϕ and length of the word $\bar{\beta}$. By the definition of γ , the number δ can be chosen in such a way that $\gamma < b/40$ (the numbers a and b were defined in Sec. 2, when the diffeomorphisms g_0 and g_1 were being defined, and this numbers can be considered to be any sufficiently close to 1 and 0, respectively, but preliminary fixed numbers).

Note that there exist arches $W^+, W^- \subset S^1$, whose lengths are not less than $1/4 - \delta$, such that the mapping f_ω expands the arch W^+ (with an expansion constant not less than $a - \delta$) and contracts the arch W^- for any sequence ω with $\omega_0 = 1$, where the symbol ω_0 stands at the zero position. By definition, put

$$P = \{p \in S^1 \mid p \text{ is the attractor of the mapping } f_\omega, \omega \in \Sigma^2, \omega_0 = 1\},$$

$$Q = \{q \in S^1 \mid q \text{ is the repeller of the mapping } f_\omega, \omega \in \Sigma^2, \omega_0 = 1\}.$$

Note that the values of $\text{diam}P$ and $\text{diam}Q$ are of order δ , and they are not more than $\gamma = K\delta^\beta$ if δ is sufficiently small. Now, when the main notations are introduced, we can formulate and prove main lemmas.

7.2. Item (6.c): main lemmas. By definition, put $S = [1/(b - \delta)]$, where $[\cdot]$ denotes the integer part. Note that if δ is sufficiently small, then S does not depend on the choice of δ .

We need the following lemma, which is a generalization of Lemma 3.3 from the paper [5]:

Lemma 8. *Let $\alpha = \alpha_{-n} \dots \alpha_{n-1}$ be a word, and $\phi_1, \phi_2 \in S^1$ be two distinct points. Then there exists a word*

$$\bar{\beta} = \beta_{-m} \dots \beta_{-n-1} \alpha_{-n} \dots \alpha_{n-1} \beta_n \dots \beta_{m-1}$$

such that the words $\beta_{-m} \dots \beta_{-n-1}$ and $\beta_n \dots \beta_{m-1}$ can not have less than T zeros in a row and

$$d_{S^1}(V_{\pm}[\bar{\beta}](\phi_1), V_{\pm}[\bar{\beta}](\phi_2)) \geq 2b.$$

Hereinafter, if no additional remarks are made, we assume that distance between two sets in the circle is length of the minimal arch connecting the points of this sets.

Remark 3. In general, the proof of Lemma 8 is similar with the proof of Lemma 3.3 from the paper [5]. All changes in this proof are connected only with the restriction on a number of zeros in the words from the lemma. We give only an outline of the proof making an accent on necessary changes.

Proof. We construct the word $\beta^l = \beta_{-l} \dots \beta_{l-1}$ inductively starting from the word α and adding by turns from one or another end of the word either $ST + 1$ zeros or one unit and ST zeros. We follow the algorithm described below. When the algorithm stops, our construction is completed (the reasoning is by induction on l). The algorithm consists of two steps.

Step 1. By definition, put

$$M_{\pm l} = \min_{\omega \in C_{\beta^l}} d_{S^1}(\bar{f}_{\pm l}[\omega](\phi_1), \bar{f}_{\pm l}[\omega](\phi_2)).$$

Check the following conditions:

$$M_l > 3b \quad (\text{B1}) \quad \text{and} \quad M_{-l} > 3b. \quad (\text{B2})$$

If both conditions hold, then the algorithm stops, further we prove that in this case the constructed word satisfies the required conditions. If at least one of conditions (B1), (B2) is violated, we go to Step 2.

Step 2. By definition, put

$$W_{\pm l} = V_{\pm}[\beta^l](\phi_1) \cup V_{\pm}[\beta^l](\phi_2).$$

Check the following conditions:

$$W_l \subset W^+ \quad (\text{C1}) \quad \text{and} \quad W_{-l} \subset W^-. \quad (\text{C2})$$

If condition (C1) (condition (C2), respectively) holds, we add from the right (from the left, respectively) one unit and ST zeros, and if it is violated, we add from the right (from the left, respectively) $ST + 1$ zeros. Then, we return to Step 1 again. Denote by \mathcal{M} the set of all l for which we return to Step 1.

In fact, it is proved in Proposition 3.1 from the paper [5] that if the algorithm stops, then the word $\bar{\beta}$ constructed by the algorithm satisfies all conditions of Lemma 8.

In order to finish the proof of Lemma 8, it is enough to show just that the described algorithm stops after a finite number of steps. Suppose the contrary, i.e., we have constructed some growing sequence of (symmetrical) words β^l , which defines the two-sided sequence ω . By definition, put

$$\begin{aligned} \phi_{jl}^\pm &= \bar{f}_{\pm l}[\omega](\phi_j), \quad j = 1, 2; \\ \delta_l^\pm &= d_{S^1}(\phi_{1l}^\pm, \phi_{2l}^\pm). \end{aligned}$$

By the construction of the sequence ω , $\phi_{jl}^\pm \in V_\pm[\beta^l](\phi_j)$. Hence, by Lemma 7 and by the definition of the numbers $M_{\pm l}$,

$$\delta_l^\pm - 2K\delta^\beta \leq M_{\pm l} \leq \delta_l^\pm.$$

By definition, put

$$W_l^\pm = V_\pm[\beta^l](\phi_1) \cup V_\pm[\beta^l](\phi_2).$$

Note that $W_l^\pm = W_{\pm l}$. To continue the proof of Lemma 8, we need the following statement:

Proposition 1. *There exists a number $m \in \mathcal{M}$ such that $\delta_m^\pm > 1/16$.*

Note that Proposition 1 implies that the described algorithm stops after a finite number of steps. Indeed, the inequality $M_{\pm m} > 3b$ holds for the number $m \in \mathcal{M}$ from Proposition 1 (of course, if b is sufficiently small compared to $1/16$).

Proof of Proposition 1. The diffeomorphism $f_{\sigma^l(\omega)}$ for $\omega_l = 0$ maps any point ϕ to some point of the arch $[\phi + b - \delta, \phi + b + \delta]$, and maps any arch of length λ into an arch of length $\lambda' \in ((1 - \delta)\lambda, (1 + \delta)\lambda)$. The diffeomorphism $f_{\sigma^l(\omega)}$ for $\omega_l = 0$ maps any arch of length λ that is contained in W^+ into an arch of length $\lambda' \in ((a - \delta)\lambda, (a + \delta)\lambda)$. And, if $\omega_l = 1$, then $\phi_{1l}^+, \phi_{2l}^+ \in W^+$, by construction.

Let us show that if $\delta_l^+ < 1/8$ ($1/8$ is approximately one half of the arch W^+), then the sequence ω can not have more than $(TS + 1)(S + 1) + TS$ zeros in a row after ω_l . Indeed, we apply the mappings δ -close to the rotations by angles

$(TS + 1)b, (2TS + 2)b, \dots, (TS + 1)(S + 1)b$; hence, one of this rotations maps the shortest one of the arches $(\phi_{1l}^+, \phi_{2l}^+)$ into the arch W^+ (since W^+ is sufficiently large). Suppose that it is the rotation by angle $(TS + 1)\ell b$. If $l - 1 \in \mathcal{M}$, then $\omega_{l+(TS+1)\ell b} = 1$, by construction. Since, it is possible that $l - 1 \notin \mathcal{M}$, it is necessary to take into consideration no more than ST zeros required to "get to" the element of \mathcal{M} , i.e., to go to Step 1.

Hence,

$$\delta_{l+(TS+1)(S+1)+TS}^+ > (a - \delta)(1 - \delta)^{(TS+1)(S+1)+TS} \delta_l^+ \quad \text{for } \delta_l^+ < 1/8. \quad (7.2)$$

If δ is sufficiently small, then $a - \delta > 1$. Since $(1 - \delta)^{(TS+1)(S+1)+TS}$ is close to 1,

$$(a - \delta)(1 - \delta)^{(TS+1)(S+1)+TS} > 1$$

if δ is sufficiently small. Therefore, the sequence $\{\delta_n^+\}$ contains an element $\delta_n^+ \geq 1/8$. Let us prove that

$$\delta_k^+ > 1/16 \quad \text{for all } k > n. \quad (7.3)$$

Suppose the contrary, i.e, there exists a number $k > n$ such that $\delta_k^+ \leq 1/16$. Choose a number $t \in [n, k)$ such that

$$\delta_t^+ \geq 1/8, \quad \delta_p^+ < 1/8 \quad \text{for all } p \in (t, k].$$

Hence, from relation (7.2) it follows that $k < t + (TS + 1)(S + 1) + TS + 1$. Indeed, since $\delta_{t+1}^+ < 1/8$,

$$\begin{aligned} \delta_{t+(TS+1)(S+1)+TS+1}^+ &> (a - \delta)(1 - \delta)^{(TS+1)(S+1)+ST} \delta_{t+1}^+ \geq \\ &\geq (a - \delta)(1 - \delta)^{(TS+1)(S+1)+ST+1} \delta_t^+ > 1/8. \end{aligned}$$

The last inequality holds, since $\delta_t^+ \geq 1/8$, and number δ can be chosen so small that

$$(a - \delta)(1 - \delta)^{(TS+1)(S+1)+ST+1} > 1.$$

Thus, $k < t + (TS + 1)(S + 1) + TS + 1$. Similar reasoning shows that there can be no ones among elements of ω_l for $t + 1 \leq l \leq k$. But then,

$$\delta_k^+ \geq (1 - \delta)^{(TS+1)(S+1)+2} 1/8 > 1/16$$

(for a sufficiently small δ), and we get a contradiction with the choice of k . The derived contradiction proves inequality (7.3).

Similarly, it can be proved that there exists a number n' such that $\delta_k^- > 1/16$ for $k > n'$. Our reasoning implies that we can assume $k, k' \in \mathcal{M}$. It proves Proposition 1, and, hence, Lemma 8. \square

\square

Consider the maximal arches \bar{W}^+ and \bar{W}^- such that

$$N(3\gamma, \bar{W}^+) \subseteq W^+ \quad \text{and} \quad N(3\gamma, \bar{W}^-) \subseteq W^-.$$

We assume γ to be sufficiently small, hence,

$$Q \subset \bar{W}^+ \quad \text{and} \quad P \subset \bar{W}^-.$$

We need the following lemma. It is similar with Lemma 3.4 from the paper [5]: we added only one new item, item (9.a), and slightly strengthened items (9.b) and (9.c). We give only an outline of the proof of Lemma 9, emphasizing the necessary changes. In addition to it, note that above-mentioned Lemma 3.4 contains one more item, which we do not need, that is why we omit it.

Lemma 9 (on distortion of arches). *Suppose that we are given an arch $J \subset S^1$ and a word $\bar{\alpha} = \alpha_{-n} \dots \alpha_0 \dots \alpha_{n-1}$. Then there exist words*

$$\bar{\beta} = \beta_{-m} \dots \beta_{-n-1} \alpha_{-n} \dots \alpha_0 \dots \alpha_{n-1} \beta_n \dots \beta_{m-1}$$

and

$$\bar{\beta}' = \beta'_{-m'} \dots \beta'_{-n-1} \alpha_{-n} \dots \alpha_0 \dots \alpha_{n-1} \beta'_n \dots \beta'_{m'-1}$$

such that

(9.a) *the words that were added to the word $\bar{\alpha}$ can not contain less than T zeros in a row;*

(9.b) *if $\omega \in C_{\bar{\beta}}$, then*

$$\bar{f}_m[\omega](J) \subset \bar{W}^- \quad \text{and} \quad W^- \subset \bar{f}_{-m}[\omega](J), \quad (7.4)$$

$$|(\bar{f}_m[\omega])'|_J| < 1, \quad |(\bar{f}_{-m}[\omega])'|_{(\bar{f}_{-m}[\omega])^{-1}(W^-)}| > 1; \quad (7.5)$$

(9.c) *if $\omega' \in C_{\bar{\beta}'}$, then*

$$\bar{f}_{-m'}[\omega'](J) \subset \bar{W}^+ \quad \text{and} \quad W^+ \subset \bar{f}_{m'}[\omega'](J), \quad (7.6)$$

$$|(\bar{f}_{-m'}[\omega'])'|_J| < 1, \quad |(\bar{f}_{m'}[\omega'])'|_{(\bar{f}_{m'}[\omega'])^{-1}(W^+)}| > 1. \quad (7.7)$$

Proof. Construct the word $\bar{\beta}$ with properties (9.a) and (9.b). Denote by ϕ_1 and ϕ_2 the ends of J . By Lemma 8, the word $\bar{\alpha}$ can be transformed into a word $\bar{\beta}_1 = \beta_{-k_1} \dots \beta_{k_1-1}$ in such a way that it would satisfy the statement of Lemma 8, i.e., the distances between the sets $V_{k_1}(\phi_1)$ and $V_{k_1}(\phi_2)$, and between the sets $V_{-k_1}(\phi_1)$ and $V_{-k_2}(\phi_2)$ are not less than $2b$.

By definition, put

$$X_{-l} := \bigcap_{\omega \in \Gamma_l} \bar{f}_{-l}[\omega](J) \quad \text{for } l \geq k_1,$$

$$Y_l := \bigcap_{\omega \in \Gamma_l} \bar{f}_l[\omega](S^1 - J) \quad \text{for } l \geq k_1;$$

i.e., X_{-l} is the interval between the sets $V_{-l}(\phi_1)$ and $V_{-l}(\phi_2)$ contained in the images of the arch J by the mapping $\bar{f}_{-l}[\omega]$, where ω "runs through" Γ_l ; and Y_l is the interval between $V_l(\phi_1)$ and $V_l(\phi_2)$ contained in the images of the arch $S^1 - \bar{J}$ by the diffeomorphism $\bar{f}_l[\omega]$, where ω "runs through" the set Γ_l (we denote by \bar{J} the closure of the arch J and by the symbol " $-$ " the set difference).

We transform the word $\bar{\beta}_1$ into the word $\bar{\beta}_2 = \beta_{-k_2} \dots \beta_{k_2-1}$ in such a way that it satisfies item (9.a) and inclusions

$$Q \subset Y_{k_2} \quad \text{and} \quad P \subset X_{-k_2}.$$

For this purpose, we add by induction (like in Lemma 8) symbols both from the left and from the right. Let l be the induction parameter, the case $l = k_1$ is the induction base.

Check two following conditions:

$$Q \subset Y_l \quad (\text{D1}) \quad \text{and} \quad P \subset X_{-l} \quad (\text{D2}).$$

If both conditions hold, the construction is completed. Otherwise, we do the following: if condition (D1) (condition (D2), respectively) holds, then we add from the right (from the left) $ST + 1$ ones, otherwise, we add from the right (from the left) $ST + 1$ zeros.

Let us show that this algorithm stops after a finite number of iterations.

Proposition 2. *If condition (D1) or (D2) holds on some iteration, then it will be satisfied up to the end of the construction.*

The essence of this proposition can be formulated as follows: addition of any number of ones can not "hurt" this conditions. The proof of Proposition 2 (as well as its formulation) is a word-by-word repetition of the proof of Proposition 3.4 from the paper [5], that is why we omit it.

Proposition 3. *Each of the conditions (D1) and (D2) holds on some iteration.*

Proof of Proposition 3. Suppose that condition (D1) never holds; the case of condition (D2) can be treated in the same way. The algorithm described above defines a certain sequence ω , and, by Proposition 2, $\omega_l = 0$ for $l \geq k_1$. Hence, the mappings $f_{\sigma^l(\omega)}$ are close to the rotation by angle b for $l \geq k_1$. In addition, by the choice of k_1 , the distance between the sets $V_{k_1}(\phi_1)$ and $V_{k_1}(\phi_2)$ is not less than $2b$. Hence,

$$\text{diam}(\bar{f}_{k_1}[\omega](S^1 - \bar{J})) \geq 2b.$$

But then, if δ is sufficiently small, one of the arches

$$\bar{f}_{k_1}[\omega](S^1 - \bar{J}), \bar{f}_{k_1+ST+1}[\omega](S^1 - \bar{J}), \dots, \bar{f}_{k_1+(S+1)ST+S+1}[\omega](S^1 - \bar{J})$$

covers the set Q in such a way that the distance from Q to its ends is more than γ . Let it be the arch $\bar{f}_{k_1+tST+t}[\omega](S^1 - \bar{J})$. By Lemma 7, the sets $V_{k_1+tST+t}(\phi_1)$ and $V_{k_1+tST+t}(\phi_2)$ are contained in the γ -neighborhoods of the ends of this arch. That is why $Q \subset Y_{k_1+tST+t}$, and condition (D1) holds. \square

Thus, we have constructed the word $\bar{\beta}_2$ of length $2k_2$ that satisfies the analogs of condition (9.a) from the formulation of the lemma and conditions (D1) and (D2). To finish the proof, one should repeat the reasoning from the end of the proof of Lemma 3.4 from the paper [5] with minor changes. Lemma 9 is proved. \square

Now, when Lemma 9 is proved, we can finish the proof of item (6.c).

7.3. Item (6.c): end of the proof. Note that sets of form $C_\alpha \times J \subset \Sigma^2 \times S^1$, where $J \subset S^1$ is an arch and α is a word, form a base of topology in the set $\Sigma^2 \times S^1$. Suppose that, as above, ω_3 is a periodically with period T_3 repeating word in the sequence prp_3 . Choose so large number $2m$ and so small arch J that

$$p_3 \subset C_{\omega_3 \dots \omega_3} \times J \subset N(d, p_3),$$

where the word ω_3 is repeated precisely m times before the zero position and precisely m times after. Next, we apply item (9.c) from Lemma 9. Suppose that $\bar{\beta}$ is the word from item (9.c) of Lemma 9. By ω denote an infinite sequence in which the word $\bar{\beta}$ is repeated periodically (and $\omega \in C_{\bar{\beta}}$). For any point $\phi \in S^1$ Lemma 5 can be applied to the point $(\omega, \phi) \in \Sigma^2 \times S^1$ (since property (9.a) holds). Hence, condition (7.1) holds for any point $s = (\omega, \phi)$.

The paper [5] proves that there exists a point $\phi_0 \in S^1$ such that the point $s := (\omega, \phi_0)$ is a hyperbolic periodic point of type $(1, 2)$, and conditions (7.6) and (7.7) hold. Item (6.c) is proved.

7.4. Proof of the remaining items of Lemma 6. We give only the proof of item (6.d). Note that the points p_1 , p_2 , p_3 and p_4 are preserved for the considered mild skew products G , and there was a lot of freedom in the choice of this points (indeed, only hyperbolicity and condition (5.1) on periods were required). In Subsec. 7.1–7.3 there was given a sufficiently detailed description of described in [5] procedure for construction of hyperbolic periodic points $p = (\omega, \phi)$ of different types that satisfy conditions (7.4) and (7.5), or (7.6) and (7.7), respectively, (depending on the type of the periodic point). This procedure allows to construct points of arbitrary large periods. That is why it can be assumed that the points p_1 and p_4 were initially constructed by such procedure for the step skew product G_0 and then fixed. In this case, items (6.a) and (6.b) are consequences of item (6.d). Thus, it is enough to prove only item (6.d).

Let us give a brief outline of the proof scheme of item (6.d). At first, we construct a pseudotrajectory $\omega \in \Sigma^2$ such that

$$\omega \in W^u(prs) \cap W^s(prp_4) \quad \text{and} \quad O(\omega, \sigma) \cap (prV_1 \cup prV_2) = \emptyset;$$

the sequence ω "includes" a certain subsequence of the sequence prs before the zero position, "includes" zeros from the zero position to the $(\bar{K} - 1)$ -position and "includes" a certain subsequence of the sequence prp_4 after the \bar{K} -position (the number \bar{K} is an arbitrary number at the moment, further it will be chosen implicitly). Then, we prove that for the constructed sequence ω there exists a point $\phi \in S^1$ such that the trajectory of the point (ω, ϕ) "goes" from the point s to the point p_4 . Next, we apply Proposition 5 (an analog of Lemma 5) and see that the trajectory of the point $y := (\omega, \phi)$ does not intersect the cylinder neighborhoods V_1 and V_2 of the sets $O(p_1, G)$ and $O(p_2, G)$. Thus, the statement of item (6.d) holds for the point y .

Choose arbitrary numbers $\bar{K} \in \mathbf{Z}$ and $m \in \mathbf{N}$. We say that two sequences $\omega = \{\beta_k\}_{k \in \mathbf{Z}}$ and $\bar{\omega} = \{\bar{\beta}_k\}_{k \in \mathbf{Z}}$ coincide on the interval $[\bar{K} - m, \bar{K} + m - 1]$ if the relation

$$\beta_k = \bar{\beta}_k \quad \text{for } \bar{K} - m \leq k \leq \bar{K} + m - 1 \quad (7.8)$$

holds. We need the following statement that is, formally, a generalization of Lemma 7 on errors. Its proof is trivial.

Proposition 4. *If the sequences $\omega, \bar{\omega} \in \Sigma^2$ satisfy relation (7.8), then the inequality*

$$d_{S^1}(\bar{f}_{\bar{K} \pm m}[\omega](\bar{\phi}_1), \bar{f}_{\bar{K} \pm m}[\bar{\omega}](\bar{\phi}_2)) \leq \gamma \quad (7.9)$$

holds, where γ is the constant from Lemma 7, and points $\bar{\phi}_1, \bar{\phi}_2 \in S^1$ are defined by the equalities

$$\bar{\phi}_1 := (\bar{f}_{\bar{K}}[\omega])^{-1}(\phi) \in S^1, \quad \bar{\phi}_2 := (\bar{f}_{\bar{K}}[\bar{\omega}])^{-1}(\phi) \in S^1. \quad (7.10)$$

Recall that the number T is defined by equality (5.2). Denote by t_p the period of the point p_4 and by t_s the period of the point s . Note that $t_s > T$. We can assume that $p_4 = (\alpha^p, \phi_p)$, and a word $\alpha_1^p \dots \alpha_{t_p}^p$ is repeated periodically in the sequence α^p in such way that the symbol α_1^p stands at the zero position of the sequence α^p . Recall that the points s and p_4 are repellers on the fibres. In Subsec. 6.3, we chose the maximal arch W^+ such that

$$N(3\gamma, \bar{W}^+) \subset W^+.$$

Since the point $p_4 = (\alpha^p, \phi_p)$ is periodic,

$$\bar{f}_{t_p}[\alpha^p](\phi_p) = \phi_p.$$

From the construction of the point s (by item (9.c) from Lemma 9) it follows that the analogs of relations (7.6) and (7.7) hold for the sequence $\omega^s = prs$,

the arch J and the certain number m_s (defined in Lemma 9). Hence, if $\bar{s} = (\alpha^s, \phi_s) = G^{m_s}(s)$, and a word $\alpha_1^s \dots \alpha_{t_s}^s$ is repeated periodically in the sequence α^s in such way that the symbol $\alpha_{t_s}^s$ stands at the (-1) -position of the sequence α^s , then the relations

$$\phi_s \subset \bar{W}^+ \subset W^+, \quad \bar{f}_{-t_s}[\alpha^s](W^+) \subset \bar{W}^+ \subset W^+, \quad |(\bar{f}_{-t_s}[\alpha^s])'|_{W^+} < 1$$

hold. This relations mean that the arch (α^s, W^+) is contained in the repelling domain of the point \bar{s} with respect to the mapping G^{t_s} , i.e., the repelling domain on fibres of the point (α^s, ϕ_s) (the repelling domain of the point ϕ_s for the restriction of the mapping G^{t_s} on the set (α^s, S^1)) contains the arch W^+ .

Since the point $\bar{s} = (\alpha^s, \phi_s)$ is periodic,

$$\bar{f}_{-t_s}[\alpha^s](\phi_s) = \phi_s.$$

Choose a set Δ_p which is a neighborhood of the point p_4 such that if

$$O_+(p, G^{t_p}) \subset \Delta_p$$

for some point p , then $p \in W^s(p_4)$.

Assume that the sequence ω is such that

- (1) it includes a word $\alpha_1^s \dots \alpha_{t_s}^s$ from the $(-t_s)$ -position to the (-1) -position, and this word is further periodically repeated in the subsequence $\omega_{k < 0}$ (the sequence ω can be considered as the mapping $\omega : \mathbf{Z} \mapsto M$, then $\omega|_A$ is the restriction of the mapping ω to a set A);
- (2) it includes a word $\alpha_1^p \dots \alpha_{t_p}^p$ from the \bar{K} -position to the $(\bar{K} + t_p - 1)$ -position, and this word is further periodically repeated in the subsequence $\omega|_{k \geq \bar{K}}$, where $\bar{K} \in \mathbf{N}$ is a certain number, which will be chosen later.

Lemma 10. (10.a) *For any number $m \in \mathbf{N}$ there exist a point ϕ_ω^m and an arch J_ω^m such that*

$$\phi_\omega^m \in N(\gamma, \phi_p); \quad (7.11)$$

$$\bar{W}^+ \subset N(\gamma, J_\omega^m) \quad \text{and} \quad J_\omega^m \subset N(\gamma, \bar{W}^+) \subset W^+; \quad (7.12)$$

$$d_{S^1}(\bar{f}_{wt_p}[\sigma^{\bar{K}}(\omega)](\phi_\omega^m), \phi_p) \leq \gamma \quad \text{for } 0 \leq w \leq 2m; \quad (7.13)$$

$$\bar{f}_{-wt_s}[\omega](J_\omega^m) \subset N(\gamma, \bar{f}_{-wt_s}[\alpha^s](\bar{W}^+)) \subset W^+ \quad \text{for } 0 \leq w \leq 2m. \quad (7.14)$$

- (10.b) *If ϕ_ω is one of the limit points of the sequence ϕ_ω^m , and an arch J_ω is a "limit arch" of the sequence J_ω^m (the meaning of this term will be clarified in the proof of the lemma), then the relations*

$$(\sigma^{\bar{K}}(\omega), \phi_\omega) \in W^s(p_4), \quad (\omega, J_\omega) \subset W^u(\bar{s}) \quad (7.15)$$

hold.

Proof. We start from the proof of item (10.a). Choose an arbitrary number $m \in \mathbf{N}$ and an arbitrary number $0 \leq k < m$.

Put $\bar{L} = mt_p$. Consider the sequence α^p . Next, we apply Proposition 4 to the "interval" $[\bar{L} - (m - k)t_p, \bar{L} + (m - k)t_p - 1] = [kt_p, 2mt_p - kt_p - 1]$ and sequences $\sigma^{\bar{K}}(\omega)$ and α^p , which coincide on this interval, by construction. Put $\phi_\omega^m = (\bar{f}_{mt_p}[\sigma^{\bar{K}}(\omega)])^{-1}(\phi_p)$. By construction,

$$\bar{f}_{t_p}[\alpha^p](\phi_p) = \phi_p.$$

Hence, by inequalities (7.9) and (7.10),

$$d_{S^1}(\bar{f}_{kt_p}[\sigma^{\bar{K}}(\omega)](\phi_\omega^m), \bar{f}_{kt_p}[\alpha^p](\phi_p)) \leq \gamma, \quad (7.16)$$

$$d_{S^1}(\bar{f}_{2mt_p - kt_p}[\sigma^{\bar{K}}(\omega)](\phi_\omega^m), \bar{f}_{2mt_p - kt_p}[\alpha^p](\phi_p)) \leq \gamma \quad (7.17)$$

for any $0 \leq k < m$. Next, we set $k = 0$ in inequality (7.16) and get inclusion (7.11). Inequalities (7.13) for $w \neq m$ follow from inequalities (7.16) and (7.17). Inequalities (7.13) for $w = m$ hold, by construction.

Put $\bar{L} = -mt_s$. Next, we apply Proposition 4 to the "interval"

$$[\bar{L} - (m - k)t_s, \bar{L} + (m - k)t_s - 1] = [-2mt_s + kt_s, -kt_s - 1]$$

and the sequences ω and α^s , which coincide on this interval, by construction. Put $V_k^+ = \bar{f}_{-kt_s}[\alpha^s](\bar{W}^+)$ and $J_\omega^m = (\bar{f}_{-mt_s}[\omega])^{-1}(V_m^+)$. It is clear that the set J_ω^m is an arch. By construction, the set (α^s, W^+) is contained in the repelling domain of the point \bar{s} , and, moreover,

$$\bar{f}_{vt_s}[\alpha^s](W^+) \subset \bar{W}^+ \subset W^+ \quad \text{for } v \in \mathbf{Z}, v \leq 0.$$

By construction of the arch \bar{W}^+ ,

$$V_{-v}^+ = \bar{f}_{vt_s}[\alpha^s](\bar{W}^+) \subset \bar{W}^+ \quad \text{for } v \in \mathbf{Z}, v \leq 0.$$

Hence, by relations (7.9) and (7.10),

$$d_H(\bar{f}_{-2mt_s + kt_s}[\omega](J_\omega^m), V_{2m-k}^+) \leq \gamma, \quad (7.18)$$

$$d_H(\bar{f}_{-kt_s}[\omega](J_\omega^m), V_k^+) \leq \gamma \quad (7.19)$$

for all $0 \leq k < m$, where d_H denotes the Hausdorff distance.

Next, we set $k = 0$ in the inequality (7.19) and get inclusion (7.12). Inequalities (7.14) for $w \neq m$ follow from inequalities (7.18) and (7.19). Inequalities (7.14) for $w = m$ hold, by construction.

Let us prove item (10.b). Let ϕ_ω be a limit point of the sequence ϕ_ω^m . Then, relations (7.13) and inclusion (7.11) hold for the point ϕ_ω and an arbitrary number w . Let j_1^m and j_2^m be the ends of the arch $J_\omega^m \subset \bar{W}^+$. Then, there exists a sequence m_k such that

$$j_e^{m_k} \longrightarrow j_e \quad \text{for } e = \{0, 1\}, \quad k \rightarrow +\infty,$$

where j_1 and j_2 are some points. Let J_ω be the arch between the points j_1 and j_2 that is contained in W^+ . The arch J_ω is called the limit arch. Note that J_ω is the set of all limit points of the sequence $J_\omega^{m_k}$. The arch J_ω satisfies relations (7.14) and inclusions (7.12) for arbitrary w .

Put

$$\bar{\Delta}_p = \{p_4 + (x - p_4)/2 | x \in \Delta_p\}.$$

Since the points p_1, p_2, p_3 and p_4 were "fixed" for all mild skew products from Theorem A', we can assume that δ was chosen to be so small that

$$N(2\gamma, \bar{\Delta}_p) \subset \Delta_p.$$

By relation (7.13), the positive semitrajectory of the point $(\sigma^{\bar{K}}(\omega), \phi_\omega)$ with respect to the mapping G^{t_p} is contained in the neighborhood $\bar{\Delta}_p$ (for sufficiently small γ ; recall that the point p_4 is fixed, and the point \bar{s} is not fixed). Consequently,

$$(\sigma^{\bar{K}}(\omega), \phi_\omega) \in W_{G^{t_p}}^s(p_4); \quad \text{hence,} \quad (\sigma^{\bar{K}}(\omega), \phi_\omega) \in W^s(p_4).$$

Similarly, by relations (7.14), the negative semitrajectories of the points of the arch (ω, J_ω) belong to a small neighborhood of the arch $(pr\bar{s}, \bar{W}^+)$.

It was already noted above that the repelling domain of the point \bar{s} on the fibres with respect to the mapping G^{t_s} contains the arch W^+ , and that is why the repelling domain of the point \bar{s} with respect to the mapping G^{t_s} is sufficiently large, to be precise, it contains a subset of the form

$$(V(pr\bar{s}) \cap W_\sigma^u(pr\bar{s}), N(\gamma, \bar{W}^+)),$$

where $V(pr\bar{s})$ is a small neighborhood of the point $pr\bar{s}$ in the base, i.e., in the set Σ^2 .

That is why relations (7.14) imply the inclusion

$$(\omega, J_\omega) \subset W_{G^{t_s}}^u(\bar{s}), \quad \text{hence,} \quad (\omega, J_\omega) \subset W^u(\bar{s}).$$

Lemma 10 is proved. \square

In particular, Lemma 10 means that relations (7.15) hold for a certain point ϕ_ω and a certain arch J_ω . Let us define the symbols that are contained in the "interval" from the zero position to the $(\bar{K} - 1)$ -position in the sequence ω . Recall that in Subsec. 7.2 we introduced the number $S = [1/(b - \delta)]$, where number b is such that the diffeomorphism g_0 defined above is the rotation by the angle b . By inclusions (7.12), the arch J_ω is sufficiently large, consequently, one of the arches

$$\bar{f}_{TS+1}[\omega](J_\omega), \dots, \bar{f}_{kTS+k}[\omega](J_\omega), \dots, \bar{f}_{(S+1)TS+S+1}[\omega](J_\omega)$$

”covers” the point ϕ_p . Suppose that it happened for the arch $\bar{f}_{kTS+k}[\omega](J_\omega)$. In this case, by definition, we put

$$\bar{K} := kTS + k,$$

and define all symbols of the sequence ω in the interval from the zero position to the $(\bar{K} - 1)$ -position to be equal to zero.

By Lemma 10, $(\omega, J_\omega) \subset W^u(\bar{s})$ and $(\sigma^{\bar{K}}(\omega), \phi_p) \in W^s(p_4)$; by construction, $(\sigma^{\bar{K}}(\omega), \phi_p) \in G^{\bar{K}}(\omega, J_\omega)$; consequently,

$$(\sigma^{\bar{K}}(\omega), \phi_p) \in W^u(\bar{s}) \cap W^s(p_4).$$

In order to finish the proof of Lemma 6, we need to prove only the following statement. The proof of this statement is similar with the proof of Lemma 5, that is why we omit it.

Proposition 5. *Suppose that a word β_p is repeated periodically with period T_4 in the sequence prp_4 , $\beta_s = \alpha_1^s \dots \alpha_{t_s}^s$ is the word constructed above which is periodically repeated in the sequence prs , and θ is a word that consists of $k(TS + 1)$ zeros for $k \geq 0$. If $\omega = \dots \beta_s \dots \beta_s \theta \beta_p \dots \beta_p \dots$, then*

$$O(\omega, \sigma) \cap (prV_1 \cup prV_2) = \emptyset.$$

Next, we apply Proposition 5 to the point $y := (\sigma^{\bar{K}}(\omega), \phi_p)$ and see that the point y satisfies all conditions of item (6.d). Lemma 6 is proved. Hence, Theorems A and A' are proved too.

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